Using Redshift to Measure the Diameters of Jupiter and Saturn and the Mass of Saturn

by David H. Samuel

Have you ever wondered how we could measure the diameter and mass of the planets in our Solar System?

This article explains how I measured the diameter of both Jupiter and Saturn, and the mass of Saturn from my front yard using a spectroscope attached to my telescope. The spectroscope was a Lhires III manufactured by Shelyak Instruments with a grating of 2400 lines / mm and a slit width of 19 µm. It is a high-resolution spectroscope by amateur standards. The telescope was an old f/10 14" (355 mm) Meade LX200 Schmidt Cassegrain Telescope (SMT), but you could use a smaller or larger telescope with different exposure times. I took 30 second exposures for the spectra. The Lhires III works best at f/10 and was originally designed to be used with an 8" Celestron SMT.

All images relating to Jupiter were taken on 19/10/2021 and images relating to Saturn were taken on 24/10/2021. Lights, darks, flats and bias frames were taken for all the planetary images. Only a single light frame was required for each calibration image because the exposures were short.

What we will be measuring is the radial velocities of Jupiter and Saturn about their axis of rotation. We will measure these velocities by calculating the redshift in our spectra. The amount of redshift will be determined by the number of pixels separating two points of our choosing in those spectra. Once we have determined their radial velocities and using the rotation periods of Jupiter and Saturn we can calculate their diameters. Going one step further, measuring the rotational velocity of Saturn's rings and applying either Kepler's Third Law or Newton's laws using the force of gravitational attraction and centripetal force describing radial acceleration, we will calculate the mass of Saturn.

Doppler shift

In the following explanation of Doppler shift, when the word light is mentioned, it refers to all electromagnetic waves, not just waves in the visible part of the spectrum.

Light has both particle (ie., photons) and wave like (i.e., wavelength, frequency) properties. Doppler shift considers the wave like properties. Light waves have a wavelength (λ) and a frequency (f) and travel at a constant velocity (c). The relationship between λ, f and c is

$$c = f\,\lambda$$

where the value of c depends on the medium through which the wave is propagating.
In a vacuum, c = 299,792,458 m/s (approximately 3 x 10^8 m/s).

Like all waves, light waves have peaks (maxima) and troughs (minima). The colours we see are determined by the wavelength of light we are looking at (i.e. the distance between successive peaks or troughs).

The Redshift is the apparent shift in wavelength caused by the Doppler effect. It is represented by the lower case letter z.

If the light source is moving away from us, we observe the peaks and troughs at a slower rate and they look as if they are further apart from each other. We see this light as having a lower frequency, longer wavelength, and being redder than what the source is emitting. We say that this light has been shifted towards the red end of the spectrum (redshift). In this case, z will have a positive value. See Figure 1.

If the light source is moving towards us, we observe the peaks and troughs at a faster rate and they look as if they are closer to each other. We see this light as having a higher frequency, shorter wavelength, and being bluer than what the source is emitting. We say that this light has been shifted towards the blue end of the spectrum (blueshift). In this case, z will have a negative value.
See Figure 1.

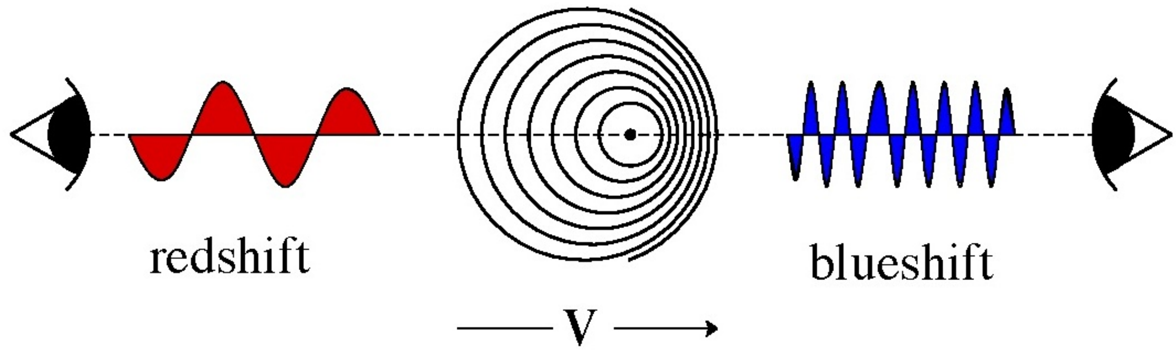

Figure 1. Redshift and blueshift

If a light source is not moving with respect to the observer, the distance between the peaks and troughs we observe are the same as what the source is emitting, which means that we see no shift (redshift or blueshift). In this case, z will have a zero value. See Figure 2.

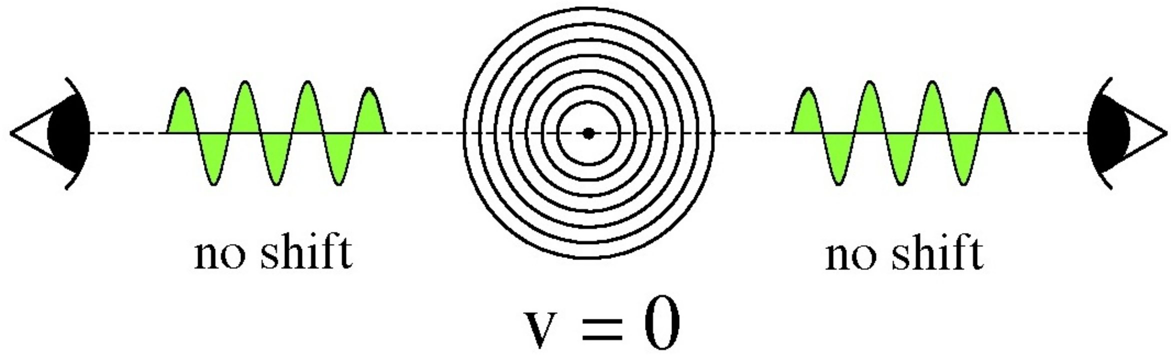

Figure 2. No shift (red or blue)

If we are observing an object moving towards us via reflected light, such as the surface of a planet being illuminated by the Sun, we will observe a double blueshift, as shown in Figure 3. This is because the object will pass the peaks and troughs from the sunlight at a faster rate and cause the peaks and troughs of the reflected light to be closer to each other than they were when originally emitted from the Sun. Thus, the reflected light will have been blue shifted relative to the light originally omitted from the Sun before it even reaches us. After reflection, because the object is also moving towards us, the peaks and troughs of the reflected light from that object will appear even closer to each other compared to after they got reflected from the object. This means that we will observe a blueshift of already blueshifted light. In this case, the value of z will indicate that the velocity of the object is twice what the actual velocity is.

The same reasoning is also true for an object moving away from us, except in that case, we will observe a double redshift instead of a double blueshift. Therefore, for reflected light, after we calculate a velocity using our measured blueshift or redshift, we will need to divide that value by two in order to find the actual velocity.

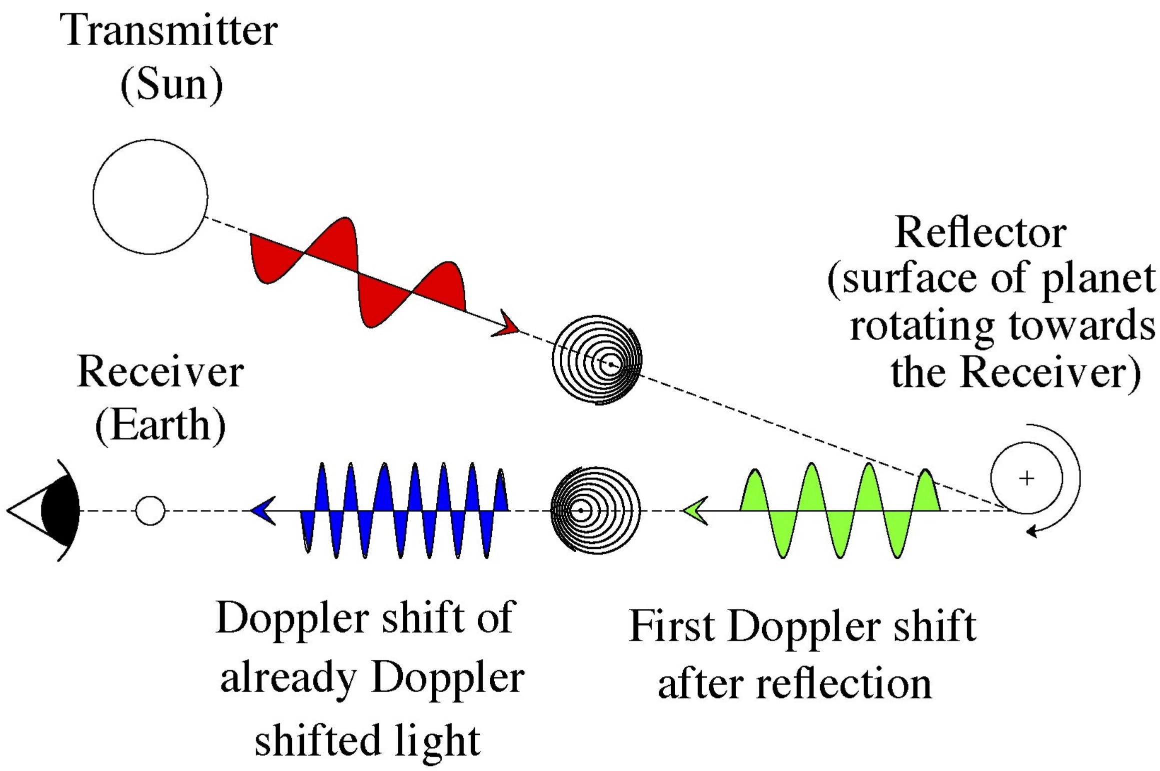

Figure 3. Doppler shift due to reflected light from an object moving towards us.

This shift of the observed wavelength (or frequency) of the light is what is known as the Doppler shift and is caused by what we call the Doppler effect. This is also true for sound waves. An Austrian physicist named Christian Doppler first proposed the Doppler effect in 1842.

The redshift or the shift in wavelength observed is in the radial (direct line of sight) direction between the observer and the source. Any relative movement between the observer and the source in any other direction does not effect the Doppler shift in any way.

The following equation for redshift is for relative velocities between the observer and the source that is much less than the speed of light ($v \ll c$). If we are talking about larger relative velocities, which we will not be doing, then we must take relativistic effects into account (relativistic redshift).

In the following equations, c is the velocity of light in a vacuum, v is the relative velocity between the observer and the source, and λ is the wavelength of the light we will be measuring.

$$\text{Redshift} = z = \frac{v}{c} = \frac{\lambda_{obsv} - \lambda_{emit}}{\lambda_{emit}} = \frac{\lambda_{obsv}}{\lambda_{emit}} - 1$$

or

$$1 + z = \frac{\lambda_{obsv}}{\lambda_{emit}} = 1 + \frac{v}{c} \quad \text{for } v \ll c.$$

Note that the redshift is a ratio and has no units.

Jupiter

Before taking any spectra, the image of Jupiter must be placed across the slit of the spectroscope. Only light from the telescope passing through the slit goes through the spectroscope. The rest of the image formed by the telescope is blocked from the spectroscope. The image is placed so that the slit lies across the equatorial plane of Jupiter, which is perpendicular to its axis of rotation. As Jupiter rotates on its axis, one edge (left or right side of the slit) will be rotating towards us and the other edge will be rotating away from us. The light from the edge that is rotating towards us will be blueshifted, and the light from the edge that is rotating away from us will be redshifted. Because we will only be interested in the difference in velocity between the two edges, we do not really need to know the actual redshift or blueshift values in absolute terms. We only need to know the difference between these values.

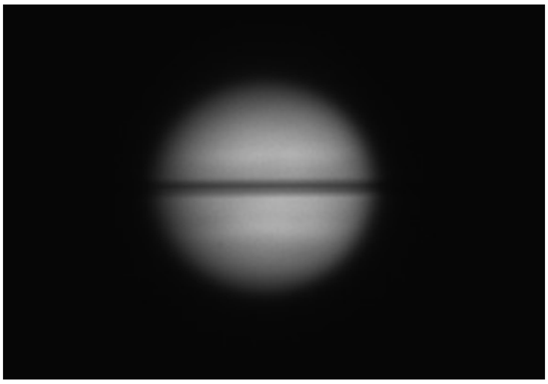

Figure 4. Image of the spectroscope slit across Jupiter's disc.

In Figure 4, the horizontal dark black line is the slit. It looks black because that is where the light has gone through to the rest of the spectroscope. The rest of the image is reflected back towards the camera to help with image placement and focusing on the slit.

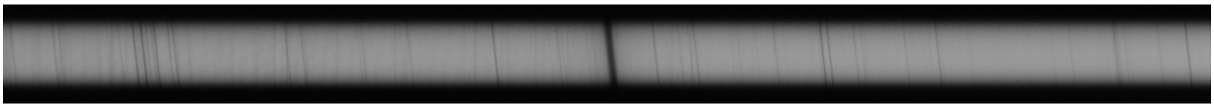

Figure 5. Jupiter's spectrum. The image was taken with a monochrome camera, so you cannot see any colours, but as is customary, I have ensured that the bluer end is on the left and the redder end is on the right.

A spectrograph of Jupiter is shown in Figure 5. There are four things you can notice even before any measurements are made.

1) All the light seen in our image is reflected sunlight. All the black vertical lines are absorption lines, which is the same as would be observed in the Sun's spectrum.

2) Most of the lines are slanted from the top left towards the bottom right. This is because the top edge is rotating towards us and is therefore blueshifted and the bottom edge is rotating away from us and is therefore redshifted.

3) A careful examination will find that there are some lines that are not slanted. They are not redshifted or blueshifted because they originate from light passing through molecules in our atmosphere (telluric lines). These lines are not used in our measurements because they do not originate from Jupiter.

4) There is a dark and wide tilted line around the centre of the image. This is the Hydrogen-alpha absorption line.

Because we need to make our measurements in units of nanometres (nm), but can only actually measure lengths in pixels, we need to work out how many nm are represented by a length of one pixel. To do this, a calibration image is taken. The calibration image is the image of a Neon-Argon calibration lamp (part of the spectroscope). This image shows spectral emmision lines at very well known wavelenghts, the values of which we either know from experience or can look up (which is what I did).

We need two lines of known wavelength and the distance between them in pixels. This allows us to calculate what is called the dispersion in nm/pixel of our imaging system. This is shown below in Figure 6 with the wavelength of the lines and distance in pixels superimposed. The camera used had a pixel size of 3.8 x 3.8 microns.

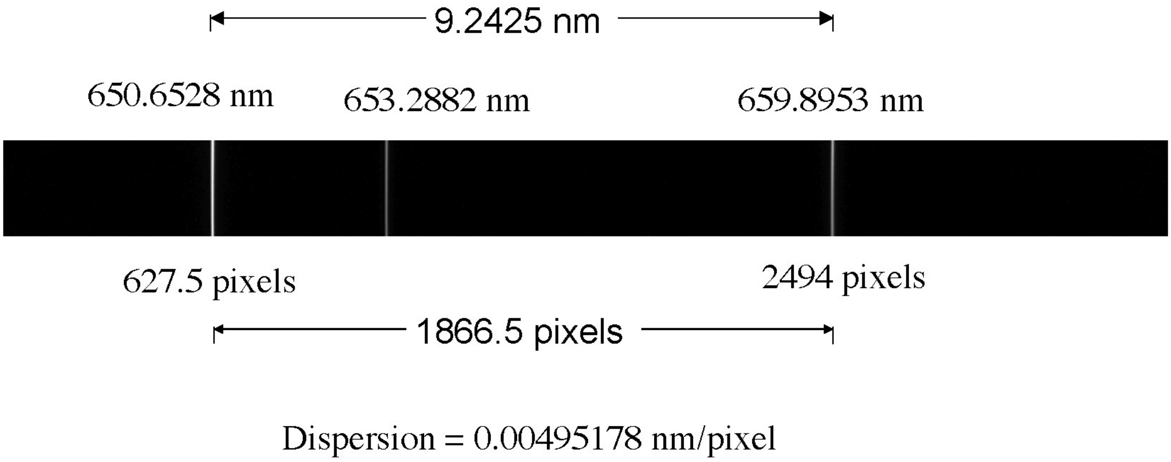

Figure 6. Neon-Argon spectral calibration lamp image for Jupiter's spectrum.

From our measurements, we find that the dispersion

$$D=\frac{659.8953-650.6528}{2494-627.5}=0.004952\pm0.000002\,nm/\,pixel\,.$$

If you are wondering why one of the measurements is not a whole number (627.5 pixels), it is because as with all of the measurements, I took the average of multiple readings to minimise errors.

Now that the dispersion is known, it can be used to convert all measurements from pixel values to nm values in the actual spectral image.

<u>Radial velocity of Jupiter about its axis of rotation</u>

An enlarged portion of Jupiter’s spectrum around the Hydrogen-alpha line is shown in Figure 7. Superimposed upon that image is shown what needs to be measured to calculate the radial velocity of Jupiter about its axis of rotation.

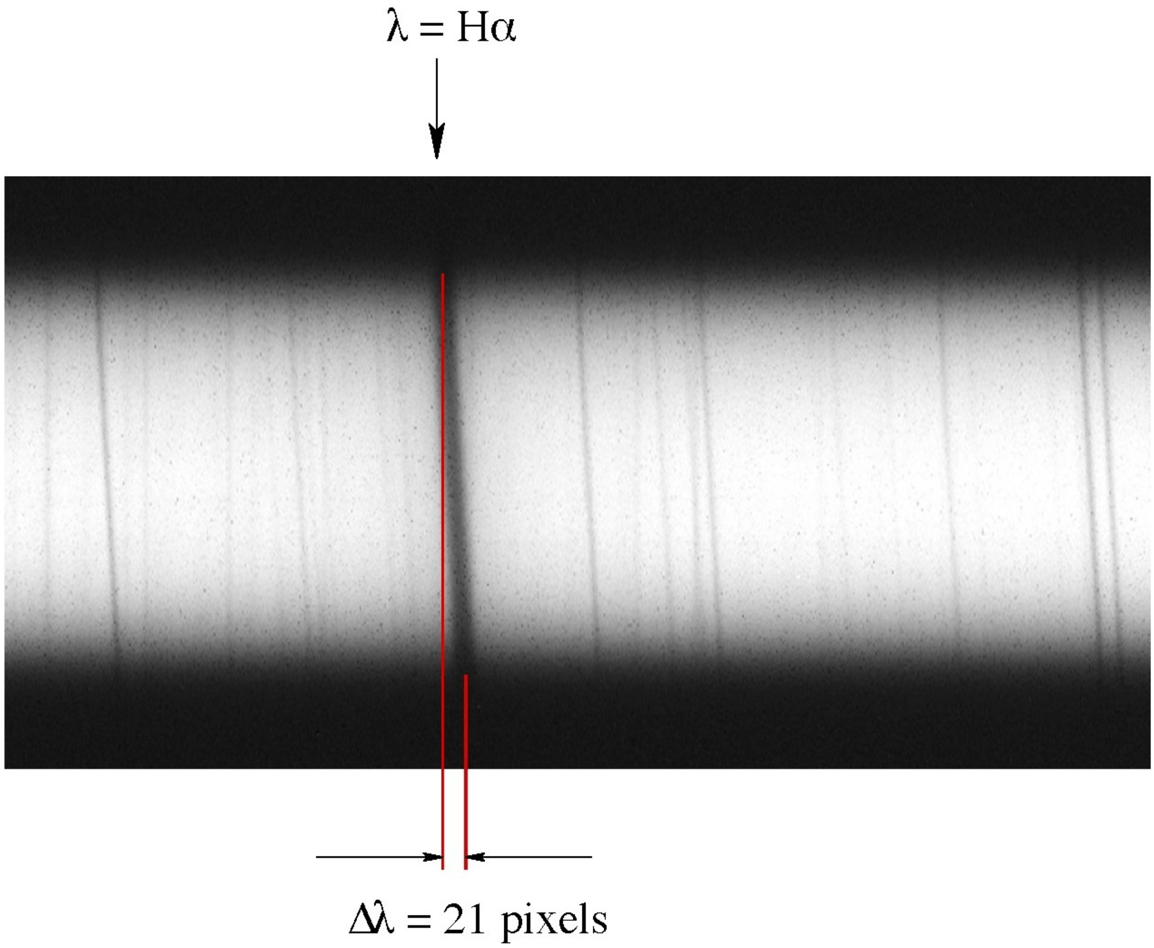

Figure 7. Enlarged portion of Jupiter's spectrum around the Hydrogen-alpha line.

I measured $\Delta\lambda$ to be 21 ± 1.4 pixels. This is converted to nm, by multiplying it by the dispersion D, which was measured earlier.

Therefore, $\Delta\lambda$ = 21 pixels x 0.004952 nm/pixel = 0.104 ± 0.007 nm.

The equation we use to calculate how fast Jupiter is rotating on its axis is the Doppler shift equation

$$\frac{\Delta\lambda}{\lambda}=\frac{v}{c} \quad \text{or} \quad v=c\frac{\Delta\lambda}{\lambda},$$

where λ = 656.282 nm (wavelength of Hydrogen-alpha light)
and c = 299,792,458 m/s (velocity of light in a vacuum) [approximately $3\text{x}10^8$ m/s].

Therefore, v = 299,792,458 m/s x 0.104 nm / 656.282 nm = 47,502 ± 3,200 m/s.

We now have to divide this by 4 to get the actual velocity of a point on Jupiter's surface at its equator, which is what we are trying to find. The reason for this is:

1) We measured the difference in velocity between the two edges of Jupiter. One edge was rotating towards us and the other edge was rotating away from us at exactly the same speed. To account for this we have to divide by 2. This is shown in Figure 8.

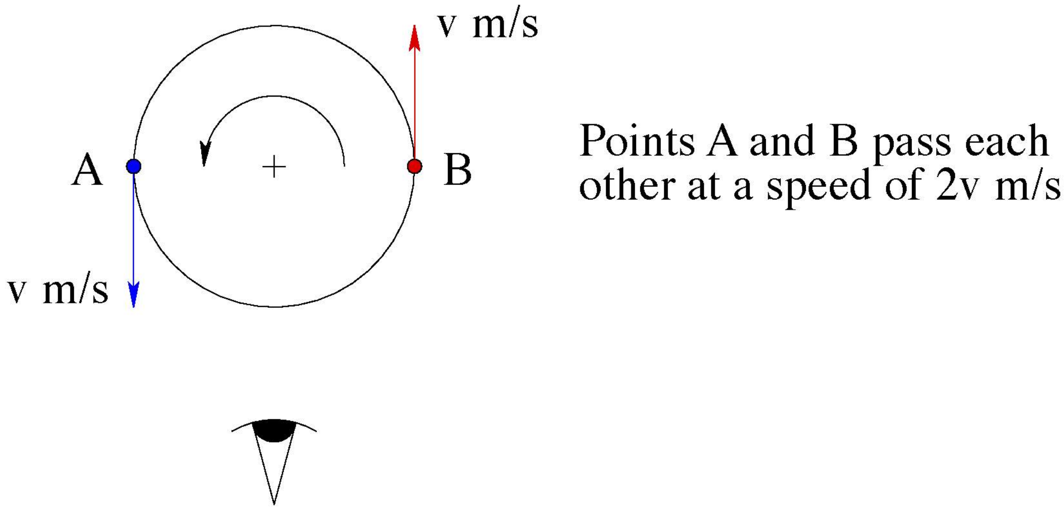

Figure 8. As Jupiter rotates on its axis, one edge rotates towards us and the other edge away from us.

2) There was a double Doppler shift in our measurements due to a reflection of sunlight from Jupiter's surface as explained earlier under the section about Doppler shift and shown in Figure 3. This means that the light that we imaged underwent a Doppler shift of already Doppler shifted light. To account for this, we have to divide by another factor of 2.

Thus the velocity of a point on Jupiter's surface at its equator (Jupiter's radial velocity about it's axis) is

$$v = \frac{47{,}502\, m/s}{4} = 11{,}875 \pm 800\, m/s \,(\text{equivalent to } 42{,}750\, km/hour).$$

Diameter of Jupiter

Using the observational results of other people[1], we know that the period of rotation of Jupiter about its axis (sidereal period) is

$$P = 9.9250 \text{ hours} = 35{,}730 \text{ seconds.}$$

Now the equatorial circumference of Jupiter is

π x J_{diam} (circumference of a circle = π x diameter), where J_{diam} is the equatorial diameter of Jupiter.

If a point on Jupiter's surface is moving at a velocity of v m/s and takes P seconds to return to its starting point, it must have traveled a distance of v x P metres, which is equal to the circumference of Jupiter.

Therefore, $\pi \times J_{diam} = v \times P$, and the diameter of Jupiter is

$$J_{diam} = \frac{vP}{\pi} \text{ metres} = 11{,}875 \times 35{,}730 / \pi = 135{,}062{,}294 \pm 9{,}100{,}000 \text{ metres}$$

or

$$J_{diam} = 135{,}100 \pm 9{,}100 \text{ km.}$$

Jupiter results compared to accepted values

The accepted value for Jupiter's radial velocity is 12.57 km/s calculated from data in a NASA website[1].
My measured value is 11.9 ± 0.8 km/s, giving a percentage uncertainty of 6.7%.

The accepted value for Jupiter's diameter is 142,984 km according to a NASA website[1].
My measured value is 135,100 ± 9,100 km, giving a percentage uncertainty of 6.7%.

Both of the NASA values lie within my calculated uncertainties.

Saturn

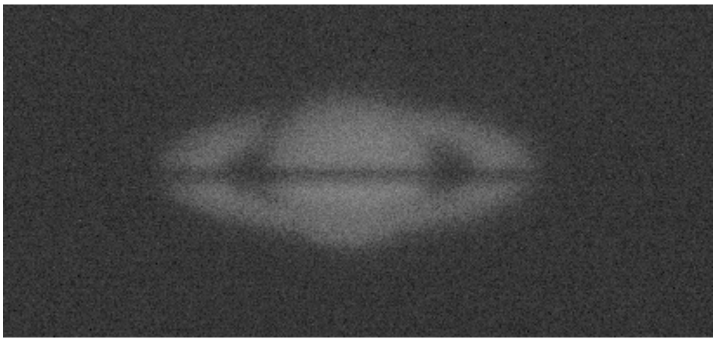

Figure 9. Image of the spectroscope slit across Saturn's disc.

Just as we did with Jupiter, we must ensure that that the image of Saturn from the telescope is placed and focused on the spectroscope slit. Again, we must try to ensure that the slit is as much as possible across Saturn's equator in order to make sure we measure the maximum velocity of the planet's rotation both towards us and away from us. In the case of Saturn, we can do this by aligning the slit along the rings as shown in Figure 9.

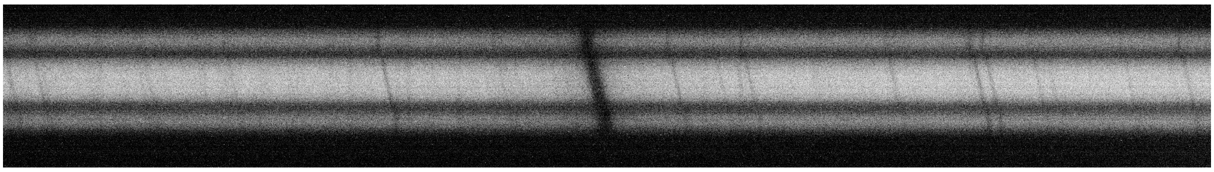

Figure 10. Saturn's spectrum. The bluer end is on the left and the redder end is on the right.

A spectrograph of Saturn is shown in Figure 10. There are six features that can be noticed even before any measurements are made.

1) All the light seen in our image is reflected sunlight. All the black vertical lines are absorption lines, which is the same as would be observed in the Sun's spectrum.

2) A careful examination will find that there are some lines that are not slanted. They are not redshifted or blueshifted because they originate from light passing through molecules in our atmosphere (telluric lines). We do not use any of these lines in our measurements because they do not originate from Saturn.

3) There is a dark and wide tilted line around the centre of the image. This is the Hydrogen-alpha absorption line.

4) There are three horizontal sections separated by wide black horizontal lines. The middle section is the spectrum of the planet. The top and bottom sections display the spectra of the ring system. The black separating lines are the gaps in between the body and the rings of Saturn. Of course, the spectral absorption lines that are visible in the body and the ring system do not exist in these gaps.

5) In the middle section, most of the lines are slanted from the top left towards the bottom right. This is because the top edge is rotating towards us and is therefore blueshifted and the bottom edge is rotating away from us and is therefore redshifted.

6) The inclination of the absorption lines that are observable in the middle section (as mentioned in (5) above), are not extended to the top and bottom sections (which is the spectrum of the ring system).

This shows us that the rings are not rotating at the same rate as the planet. In fact, a closer examination indicates that the part of the ring system close to the planet is rotating a little faster than the planet. Absorption lines in the spectrum of the part of the ring system further away from the planet are inclined slightly in the opposite direction to the inclination of the absorption lines of the planet which indicate that they are moving a little slower than the planet. This will be more observable in the enlarged view of a portion of the spectrum shown in Figure 13.

A calibration image is taken to enable the conversion of the horizontal coordinate values of pixels in Saturn's spectral image to nm. We need two lines of known wavelength and the distance between them in pixels. This allows us to calculate the dispersion in nm/pixel of our imaging system. This is shown in Figure 11 with the wavelength of the lines and distance in pixels superimposed. The camera used had a pixel size of 2.4 x 2.4 microns.

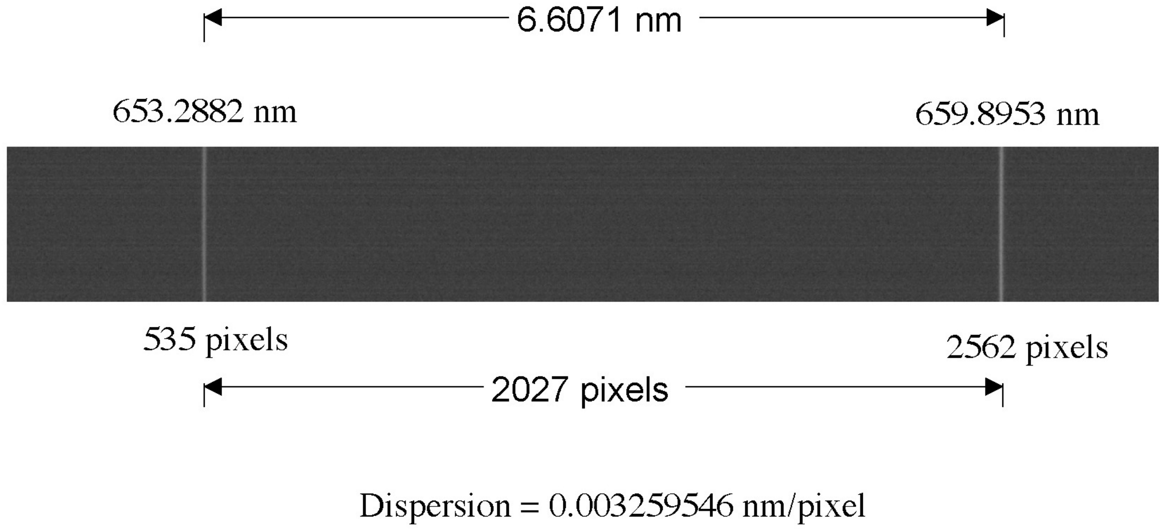

Figure 11. Neon-Argon spectral calibration lamp image for Saturn's spectrum.

From our measurements, we find that the dispersion

$$D = \frac{659.8953 - 653.2882}{2562 - 535} = 0.003260 \pm 0.000001 \text{ nm/pixel.}$$

Now that we know the dispersion, we can use that to convert all of our measurements from pixel values to nm values in our actual spectral image.

<u>Inclination of Saturn's Axis</u>

When we make our measurements from Saturn's spectrum, we go through basically the same procedures we used for Jupiter with one important exception. Jupiter has an axial tilt of only 3.13 degrees (according to a NASA website[1]). Saturn has an axial tilt of 26.73 degrees (according to a NASA website[2]). We did not take this into account in our Jupiter measurements because it would have made a negligible difference to our final results. However, for Saturn we need to take this into account.

When we calculate any velocities from our redshift values what we are measuring is the velocity directly towards us (blue shifted) or away from us (red shifted). What we are trying to find is the velocity of rotation of Saturn about its axis of rotation, which will be slightly higher in value than what we measure because the axis of rotation is not perpendicular to our line of sight. If Saturn's axis is tilted towards us by θ degrees, then all velocity measurements made must be divided by the cosine of θ (or alternatively multiplied by the secant of θ). This is shown in Figure 12.

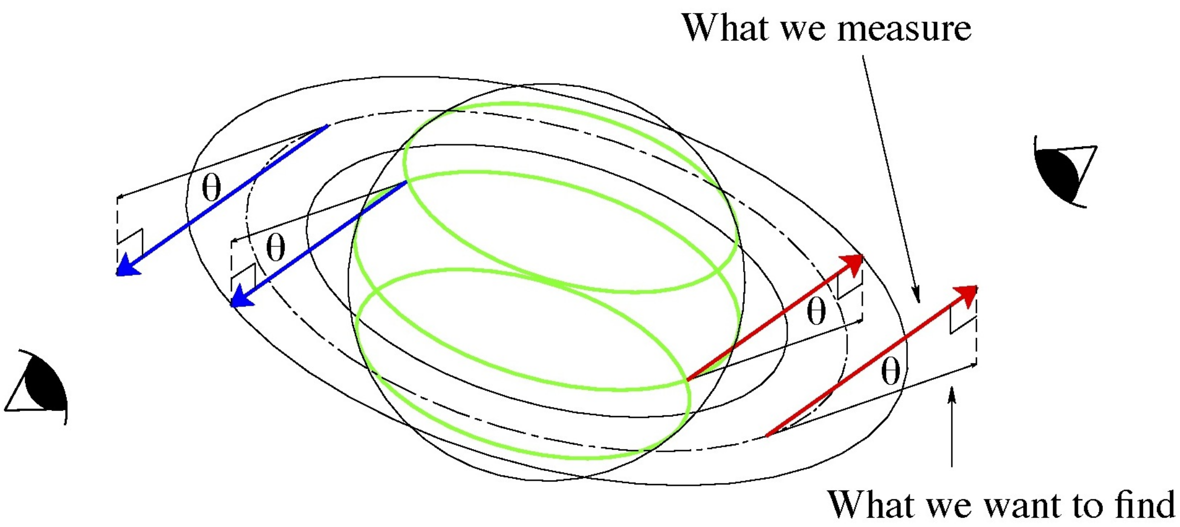

Figure 12. Difference between what we measure and what we want to find because of Saturn's inclination towards the Earth.

There are two methods we can use to find the inclination of Saturn towards the Earth. The first is by measuring the ratio of the minor and major axis of the ellipse formed by the rings in Figure 9. I did not use this method because my image was not sharp enough for me to make accurate enough measurements.

The second method is to use a Saturn ephemeris generator, which is the method I used. There is an ephemeris generator available online at the following website:

"https://pds-rings.seti.org/tools/ephem2_sat.shtml".

Using generated data from that website, we find that the inclination of Saturn towards the Earth was 19.39 degrees at the time I took the spectral images.

From Saturn's Spectral Image

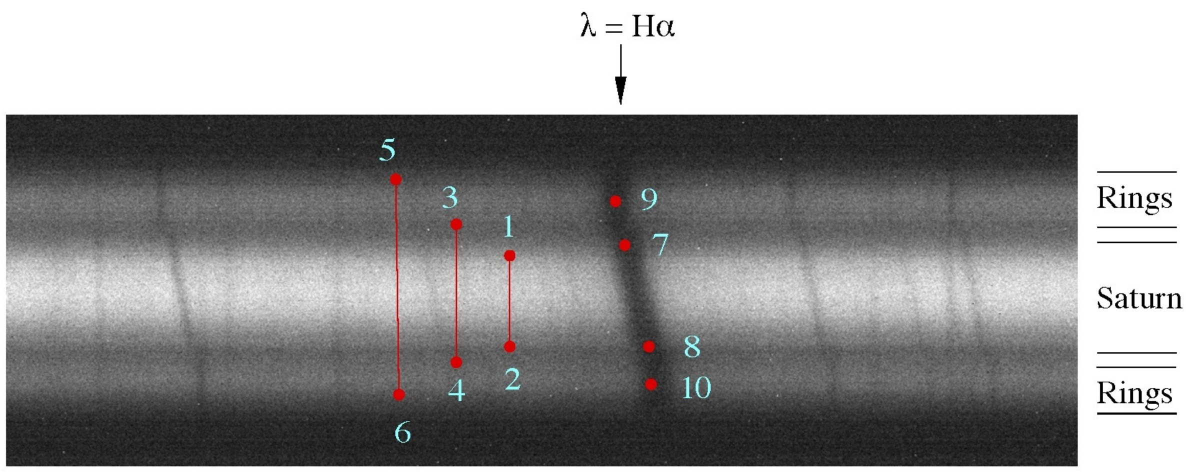

Figure 13. Enlarged portion of Saturn's spectrum around the Hydrogen-alpha line. Superimposed upon that image is shown what needs to be measured to calculate the radial velocity of Saturn about its axis of rotation, and the mass of Saturn.

Radial velocity of Saturn about its axis of rotation

I measured the horizontal distance between points 7 and 8 in Figure 13 to be 25 ± 1.4 pixels. This is the $\Delta\lambda$ in our Doppler shift equation and is converted to nm, by multiplying it by the dispersion D.

Therefore, $\Delta\lambda$ = 25 pixels x 0.003260 nm / pixel = 0.081 ± 0.005 nm.

The Doppler shift equation is $\frac{\Delta\lambda}{\lambda}=\frac{v}{c}\vee v=c\frac{\Delta\lambda}{\lambda}$ (without having corrected for axial tilt), and to correct for axial tilt we divide by cos(θ).

Thus, $v = c\,\frac{\Delta\lambda}{\lambda} / \cos(\theta)$

where λ = 656.282 nm (wavelength of Hydrogen-alpha light), c = 299792458 m/s (velocity of light) and θ = 19.39 degrees (Saturn's axial tilt).

Thus, we find that $v = (299{,}792{,}458 \text{ m/s}) \frac{0.081\,nm}{656.282\,nm} / \cos(19.39 \text{ degrees}) = 39{,}463 \pm 2{,}232$ m/s.

We now have to divide this by 4 to get the actual velocity of a point on Saturn's surface at its equator, which is what we are trying to find. The reasons for this are the same as described earlier for Jupiter.

Thus the velocity of a point on Saturn's surface at its equator (Saturn's radial velocity about it's axis) is

$$v=\frac{39{,}463\,m/s}{4}=9{,}900\pm 600\,m/s\left(\text{equivalent to about }35{,}500\,km/hour\right).$$

Saturn's diameter

Using the observational results of other people[2], we know that the period of rotation of Saturn about its axis (sidereal period) is

$$P = 10.656 \text{ hours} = 38{,}362 \text{ seconds.}$$

Now the circumference of Saturn is

π x S_{diam} (circumference of a circle = π x diameter), where S_{diam} is the diameter of Saturn.

If a point on Saturn's surface is moving at a velocity of v m/s and take P seconds to return to its starting point, it must have travelled a distance of v x P metres, which is equal to the circumference of Saturn.

Therefore, $\pi \times S_{diam} = v \times P$ or the diameter of Saturn is

$$S_{diam} = \frac{vP}{\pi} \text{ m} = 9{,}866 \times 38{,}362 / \pi = 120{,}468{,}440 \pm 6{,}800{,}000 \text{ m}$$

or

$$S_{diam} = 120{,}500 \pm 6{,}800 \text{ km.}$$

Mean diameter of Saturn's rings

I made the following measurements from the image of Saturn's spectrum:

(a) The diameter of Saturn is 114 ± 1.4 pixels (vertical distance between points 1 and 2).

(b) The inner diameter of Saturn's rings is 152 ± 1.4 pixels (vertical distance between points 3 and 4).

(c) The outer diameter of Saturn's rings is 269 ± 1.4 pixels (vertical distance between points 5 and 6).

(d) The mean diameter of Saturn's rings is 210.5 ± 2 pixels (average value of (b) and (c)).

(e) The mean diameter of Saturn's rings is also equal to 1.85 ± 0.03 planetary diameters (210.5 / 114). Since we have already calculated Saturn's diameter as being 120,468,440 m, we can now calculate the mean diameter of Saturn's rings as

S_{rings} = 120,468,440 m x 1.85 = 222,443,919 ± 13,000,000 m = 222,444 ± 13,000 km.

Please keep in mind that when we are calculating the diameter of Saturn's rings, we are only working out the diameter of the part of the ring system that we can see in our spectrograph. In reality, the rings are more extensive, but if we cannot see them in our spectrograph, we cannot take them into account in our measurements. However that will not effect our calculation of Saturn's mass because that calculation relies on the velocity of the part of the ring system for which we have measured the diameter.

Radial velocity of Saturn's rings

I measured $\Delta\lambda$ to be 47 ± 1.4 pixels (horizontal distance between points 9 and 10). To convert this to nm, we multiply it by the dispersion D.

Therefore, $\Delta\lambda$ = 47 pixels x 0.003260 nm/pixel = 0.1532 ± 0.0046 nm.

The equation we use to calculate the radial velocity of Saturn's rings is as we did before for both Jupiter and Saturn:

$$\frac{\Delta\lambda}{\lambda}=\frac{v}{c}$$

or

$$v=c\frac{\Delta\lambda}{\lambda}(\textit{without having corrected for axial tilt}).$$

and to correct for axial tilt we divide this by $\cos(\theta)$, giving us

$$v=c\frac{\Delta\lambda}{\lambda}/\cos(\theta)$$

where λ = 656.282 nm (wavelength of Hα light), c = 299792458 m/s (velocity of light) and θ = 19.39 degrees (Saturn's axial tilt).

Thus, we find that

v = (299,792,458 m/s x 0.1532 nm / 656.282 nm) / cos(19.39 degrees) = 74,190 ± 2,200 m/s.

We now have to divide this value by 4 (for the same reasons given earlier), to get the actual velocity of the part of the ring system, which is at a distance equal to the mean radius of the rings from the centre of Saturn, which is what we are trying to find.

Therefore our final measurement of the radial velocity of Saturn's rings is

v = (74,190 m/s) / 4 = 18,500 ± 600 m/s (equivalent to about 66,770 km/hour).

Saturn's mass

Saturn's rings consist of many small objects orbiting the planet. We will use the gravitational attraction between one of these small objects and the planet to calculate the mass of Saturn by using the following equation, which can be derived from Kepler's third law or from Newton's laws using the force of gravitational attraction and centripetal force describing radial acceleration. The derivations are shown in the Appendix at the end of this article.

$$M = \frac{v^2 r}{G}$$

where

M is the mass of Saturn,
v is the mean velocity of an object making up Saturn's rings,
r is the mean distance from the centre of Saturn to an object making up Saturn's rings,
G is the Gravitational constant = 6.67 x 10^{-11} Nm^2kg^{-2}

Note: $r = \frac{mean\, diameter\ of\ Saturn's\, rings}{2}$.

Thus Saturn's mass M is

M = $(18547\ m/s)^2$ x (222443919 m / 2) / 6.67 x 10^{-11} Nm^2kg^{-2} = (5.7 ± 0.4) x 10^{26} kg.

Saturn results compared to accepted values

The accepted value for Saturn's radial velocity is 9.871 km/s. I calculated this from data given in a NASA website[2].
My measured value is 9.9 ± 0.6 km/s, giving a percentage uncertainty of 5.7%.

The accepted value for Saturn's diameter is 120,536 km according to a NASA website[2].
My measured value is 120,500 ± 6,800 km, giving a percentage uncertainty of 5.7%.

The accepted value for Saturn's mass is 5.6834 x 10^{26} kg according to a NASA website[2].
My measured value is (5.7 ± 0.4) x 10^{26} kg, giving a percentage uncertainty of 7%.

All three of the NASA values lie within my calculated uncertainties.

References

All images and diagrams by the author.

[1] https://nssdc.gsfc.nasa.gov/planetary/factsheet/jupiterfact.html, Ed Grayzeck, 16 December 2021.

[2] https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturnfact.html, Ed Grayzeck, 16 December 2021.

Shelyak Instruments Lhires III User Guide, pages 38 - 39, "https://www.shelyak.com/wp-content/uploads/DC0004A-Lhires-III-User-Guide-English-1.pdf".

Spectral Lines as Distant Measurement Tools, pages 7-11, "https://robrutten.nl/rrweb/rjr-edu/exercises/assignments-spectral-lines/line-usage.pdf".

Reviewers

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Edwin Samuel – Graduate, University of New England, Australia

Appendix A - Calculation of Uncertainties in Measurements

All uncertainties in this article were handled as follows.

In the following the uncertainty off a value *u* is shown as Δu.

1) Independent uncertainties - most of the calculations involved independent uncertainties.

Adding or subtracting - uncertainties were added in quadrature

$$\Delta u = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + \cdots}$$

Multiplying or dividing - fractional uncertainties were added in quadrature

$$\frac{\Delta u}{u} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \left(\frac{\Delta z}{z}\right)^2 + \cdots}$$

2) Dependent uncertainties - an example of this is when I divided the measured redshift velocity by 4 to get the actual velocity.

Adding or subtracting - absolute uncertainties were added linearly

$$\Delta u = \Delta x + \Delta y + \Delta z + \cdots$$

Multiplying or dividing - fractional uncertainties were added linearly

$$\frac{\Delta u}{u} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z} + \cdots$$

Appendix B - Derivation of the equation used to calculate Saturn's mass

For derivation of this equation, two methods are available.

Using Kepler's third law

Kepler's third law is

$$P^2 = \frac{4\pi^2 a^3}{G(M+m)}. \qquad [1]$$

where

P is the period of orbit (time taken for the object to travel around the planet once)
M is the mass of Saturn
m is the mass of one of the objects making up Saturn's rings
a is the semi-major axis length (= r for a circular orbit)
G is the Gravitational constant = 6.67 x 10^{-11} Nm^2kg^{-2}

The mass of Saturn (M) is much larger than the mass of a single object making up its rings (m), which means that

$$M + m \approx M. \qquad [2]$$

Also, for a circular orbit

$$P = \frac{2\pi r}{v}. \qquad [3]$$

Combining equations [1] to [3] gives us

$$M = \frac{v^2 r}{G}.$$

Using Newton's laws

The centripetal force describing the radial acceleration of one of the objects making up Saturn's rings in a circular orbit around Saturn is

$$F_c = \frac{m v^2}{r}.$$

The force of gravitational attraction between Saturn and that same object making up its rings is

$$F_g = \frac{G M m}{r^2}.$$

Now since $F_c = F_g$ (the centripetal force and the gravitational force are one and the same) we get

$$\frac{m v^2}{r} = \frac{G M m}{r^2}$$

or

$$M = \frac{v^2 r}{G}.$$